

Solutions to Mathematics Workbook-1 | Quadratic Equations & Inequalities [2022]

Level - 2	Daily Tutorial Sheet 12
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140.(3) Given, α, β are roots of $x^2 - 6x - 2 = 0$

$$\therefore \alpha^2 - 6\alpha - 2 = 0$$

$$\Rightarrow \alpha^{10} - 6\alpha^9 - 2\alpha^8 = 0 \quad \dots \text{(i)}$$

$$\text{and } \beta^2 - 6\beta - 2 = 0$$

$$\Rightarrow \beta^{10} - 6\beta^9 - 2\beta^8 = 0 \quad \dots \text{(ii)}$$

On subtracting equation (i) from equation (ii), we get :

$$(\alpha^{10} - \beta^{10}) - 6(\alpha^9 - \beta^9) - 2(\alpha^8 - \beta^8) = 0$$

$$\Rightarrow \alpha^{10} - 2\alpha^8 = 6\alpha^9$$

$$\therefore \frac{\alpha^{10} - 2\alpha^8}{2\alpha^9} = 3$$

141.(6) $f(x) = (kx + 3)(x - 4)$

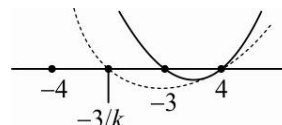
$f(x) < 0$ for $x \in (-3, 3)$ and $f(-4) > 0$

$$\Rightarrow -4 < \frac{-3}{k} \leq -3 \quad \Rightarrow \quad 4 > \frac{3}{k} \geq 3$$

$$\Rightarrow \frac{1}{4} < \frac{k}{3} \leq \frac{1}{3} \quad \Rightarrow \quad \frac{3}{4} < k \leq 1$$

$$\therefore k \in \left(\frac{3}{4}, 1 \right]$$

$$\therefore p = \frac{3}{4} \text{ and } q = 1 \quad \Rightarrow \quad 4p + 3q = 6$$



142.(9) Product of roots $= \frac{1}{5}$

Since the quadratic equation becomes negative for some x and coefficient of $x^2 > 0$,

Roots are distinct and Product $> 0 \Rightarrow$ They have same sign

$$\text{Sum of roots} = \frac{2k}{5}$$

Since we want (+ve) values of k ,

$$\text{Sum} > 0$$

$\Rightarrow$ Both roots are (+ve)

Let the 2 roots be $\frac{\alpha}{\sqrt{5}}, \frac{1}{\sqrt{5}\alpha}$ with $\alpha > 1$ and thus $\frac{\alpha}{\sqrt{5}} > \frac{1}{\sqrt{5}} > \frac{1}{\sqrt{5}\alpha} > 0$

$$\alpha > 1 \Rightarrow 0 < \frac{1}{\sqrt{5\alpha}} < \frac{1}{\sqrt{5}} \text{ and } \frac{\alpha}{\sqrt{5}} > \frac{1}{\sqrt{5}}$$

Now the given quadratic equation < 0

$$\forall x \in \left(\frac{1}{\sqrt{5\alpha}}, \frac{\alpha}{\sqrt{5}} \right)$$

And the question requires that $\left(\frac{1}{\sqrt{5\alpha}}, \frac{\alpha}{\sqrt{5}} \right)$ has exactly 1 integer

That integer can only be 1

$$\text{Thus, } 1 < \frac{\alpha}{\sqrt{5}} \leq 2$$

$$\Rightarrow \sqrt{5} < \alpha \leq 2\sqrt{5}$$

$$\frac{2k}{5} = \frac{1}{\sqrt{5}} \left(\alpha + \frac{1}{\alpha} \right)$$

$$\text{Also, } \alpha_2 > \alpha_1 > 1 \Rightarrow \alpha_2 + \frac{1}{\alpha_2} > \alpha_1 + \frac{1}{\alpha_1} > 1$$

Because $f(x) = x + \frac{1}{x}$ increases for all $x > 1$

$$\text{Or } \alpha + \frac{1}{\alpha} = \left(\sqrt{\alpha} - \frac{1}{\sqrt{\alpha}} \right)^2 + 2$$

As α increases after 1, $\sqrt{\alpha}$ increases and $\frac{1}{\sqrt{\alpha}}$ decreases

$$\Rightarrow \left(\sqrt{\alpha} - \frac{1}{\sqrt{\alpha}} \right)^2 \text{ increases}$$

$$\text{Hence, } \sqrt{5} < \alpha \leq 2\sqrt{5}$$

$$\Rightarrow \frac{6}{5} < \frac{2k}{5} = \left(\alpha + \frac{1}{\alpha} \right) \times \frac{1}{\sqrt{5}} < \frac{21}{10}$$

$$\Rightarrow 3 < k < 5.25$$

$$\Rightarrow k = \{4, 5\} \text{ because } k \in \mathbb{Z}$$

Hence, sum of all integer values = 9

143.(1) The given quadratic equation has real roots

$$\Rightarrow D \geq 0$$

$$\Rightarrow k^2 - 4(k^2 + 7k + 15) \geq 0 \Rightarrow -(3k^2 + 28k + 60) \geq 0 \Rightarrow -3 \left(k^2 + \frac{28k}{3} + 20 \right) \geq 0$$

$$\begin{aligned}
 \Rightarrow & -3 \left[k^2 + \frac{28k}{3} + \left(\frac{14}{3} \right)^2 \right] + 3 \left(\frac{14}{3} \right)^2 - 60 \geq 0 \Rightarrow \frac{16}{3} - 3 \left(k + \frac{14}{3} \right)^2 \geq 0 \\
 \Rightarrow & 3 \left[\frac{16}{9} - \left(k + \frac{14}{3} \right)^2 \right] \geq 0 \Rightarrow \left(k + \frac{14}{3} \right)^2 \leq \frac{16}{9} \\
 \Rightarrow & -\frac{4}{3} \leq k + \frac{14}{3} \leq \frac{4}{3} \\
 \Rightarrow & -6 \leq k \leq -\frac{10}{3} \quad \dots\dots\dots(1)
 \end{aligned}$$

Also: $x_1^2 + x_2^2 = (x_1 + x_2)^2 - 2x_1x_2 = k^2 - 2(k^2 + 7k + 15)$

$$\begin{aligned}
 &= -(k^2 + 14k + 30) = -(k + 7)^2 + 49 - 30 \\
 &= 19 - (k + 7)^2 \quad \dots\dots\dots(2)
 \end{aligned}$$

From (1) $k \in \left[-6, -\frac{10}{3} \right]$

$$k + 7 \in \left[1, \frac{11}{3} \right]$$

$$(k + 7)^2 \in \left[1, \frac{121}{9} \right]$$

$$\Rightarrow x_1^2 + x_2^2 \leq 18 \quad \{(k + 7)^2 \geq 1\}$$

Hence, maximum value = 18 and $x = 1$

144.(7) For the given quadratic equation to have at least 1 root > 0 , it must

(a) Have real roots: $D \geq 0 \Rightarrow (a - 10)(a - 4) \geq 0 \Rightarrow a \geq 10$ or $a \leq 4$

(b) Not have both roots ≤ 0 $\{D \geq 0 \text{ \& sum of roots } \leq 0 \text{ \& Product roots } \geq 0\}$

$$\begin{aligned}
 \Rightarrow & a \geq 10 \text{ or } a \leq 4 \quad \{D \geq 0\} \\
 & a \leq 5 \quad \{\text{sum} \leq 0\} \\
 & a \geq \frac{15}{4} \quad \{\text{Product} \geq 0\}
 \end{aligned}$$

$$\Rightarrow a \in \left[\frac{15}{4}, 4 \right]$$

(a) - (b) gives $a < \frac{15}{4}$ or $a \geq 10$

Hence, in the range $a \in [1, 100]$

a can only take integral values $\{1, 2, 3, 10, 11, \dots, 100\}$

Their sum $= \frac{3(3+1)}{2} + \frac{91}{2}(10+100) = 6 + 91 \times 55 = 5011$

145.(1) Given equation is $x^2 - x + 1 = 0$

$$\Rightarrow x = \frac{1 \pm \sqrt{1-4}}{2} \Rightarrow x = \frac{1 \pm \sqrt{-3}}{2}$$

$$x = \frac{1 \pm \sqrt{3}i}{2}$$

We know that, $\omega = \frac{-1 + \sqrt{3}i}{2}$ and $\omega^2 = \frac{-1 - \sqrt{3}i}{2}$

$$\therefore \alpha = \frac{1 + \sqrt{3}i}{2} = -\omega^2 \text{ and } \beta = \frac{1 - \sqrt{3}i}{2} = -\omega$$

$$\begin{aligned} \text{Now, } \alpha^{2009} + \beta^{2009} &= (-\omega^2)^{2009} + (-\omega)^{2009} \\ &= (-\omega^2)^{3 \times 669 + 2} - (\omega)^{3 \times 669 + 2} \\ &= -(\omega^3)^{2 \times 669} \cdot \omega^4 - (\omega^3)^{669} \cdot \omega^2 \\ &= -\omega^4 - \omega^2 \quad \left[\because \omega^3 = 1 \right] \\ &= -\omega^3 \cdot \omega - \omega^2 = -\omega - \omega^2 = 1 \quad \left[\because 1 + \omega + \omega^2 = 0 \right] \end{aligned}$$

146.(3) Let $x^2 - 5x + 4 = a \dots (i)$

$$\therefore a(y^2 + y + 1) < 2y$$

$$\Rightarrow ay^2 + y(a - 2) + a < 0 \text{ for all } y \in R$$

$$\therefore a < 0 \text{ and } D < 0$$

$$\Rightarrow a < 0 \text{ and } (a - 2)^2 - 4a^2 < 0$$

$$\Rightarrow 3a^2 + 4a - 4 > 0$$

$$\therefore a < 0 \text{ and } a < -2$$

$$\Rightarrow a < -2 \dots (ii)$$

From equations (i) and (ii), $x^2 - 5x + 4 < -2$

$$\Rightarrow x^2 - 5x + 6 < 0 \Rightarrow x \in (2, 3)$$

147.(4) Given, $\log_x y = -\frac{1}{4} \Rightarrow y = x^{-1/4} \Rightarrow x^{1/4} y = 1$

$$\Rightarrow xy^4 = 1 \dots (i)$$

$$\text{and } E = \log_x (xy^5) \left[2\log_y x - \frac{1}{2}\log_y y \right]$$

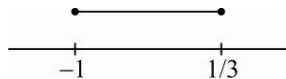
$$= \log_x (xy^4 \cdot y) - 2\log_y x + \frac{1}{2} = \log_x (xy^4) + \log_x y - 2\log_y x + \frac{1}{2}$$

$$= 0 + \left(-\frac{1}{4} \right) - 2(-4) + \frac{1}{2} = -\frac{1}{4} + 8 + \frac{1}{2} + \frac{1}{4} = \frac{33}{4} = \frac{p}{q}$$

$$\Rightarrow q \text{ is } 4.$$

148.(5) Here, $3x^2 + 2x - 1 < 0 \Rightarrow (x+1)(3x-1) < 0$

So, $(3a-2-a^2)x+2 < 0$ for some x in $\left(-1, \frac{1}{3}\right)$ and $(a^2-3a+2)x-2 > 0$ for some x in $\left(-1, \frac{1}{3}\right)$



Now, let $f(x) = -(a^2 - 3a + 2)x + 2$

$\therefore f(-1) = a^2 - 3a + 4 > 0$

Since $f(x)$ is linear in x ,

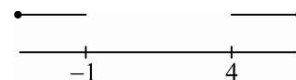
$f(-1) > 0$ and $f\left(\frac{1}{3}\right) > 0$ ensures

$f(x) > 0 \quad \forall x \in \left[-1, \frac{1}{3}\right]$

But since we want $f(x)$ to be (-ve) for some $x \in \left(-1, \frac{1}{3}\right)$, and $f(-1) > 0$,

$f\left(\frac{1}{3}\right) < 0$ becomes a necessary and sufficient condition both

and $f\left(\frac{1}{3}\right) = \frac{-(a^2 - 3a + 2)}{3} + 2 > 0 \Rightarrow a^2 - 3a + 4 > 0$



and $a^2 - 3a - 4 > 0 \Rightarrow a^2 - 3a - 4 > 0$

$\Rightarrow (a-4)(a+1) > 0$

$\therefore a < -1$ or $a > 4$ Thus, least natural number is 5.

149.(4) Sum of roots = $-b$ an odd integer

$\Rightarrow$ One root is odd & the other is even $\Rightarrow$ Only even prime natural number = 2

$\Rightarrow$ 2 is a root and thus $4 + 2b + c = 0 \Rightarrow |2b + c| = 4$

150.(0) $|x| + \left|\frac{x}{x-1}\right| = \frac{x^2}{|x-1|}$

As, $|a| + |b| = |a+b|$ only, when $ab \geq 0$

And $|x| + \left|\frac{x}{x-1}\right| = \left|x + \frac{x}{x-1}\right|$ is true only, if $x \cdot \frac{x}{x-1} \geq 0$

$\Rightarrow x \in \{0\} \cup (1, \infty)$

$\therefore$ Least value of $x = 0$

151.(1) Given $4y^2 + 4xy + x + 6 = 0$

$$\Rightarrow y = \frac{-4x \pm \sqrt{16x^2 - 16(x+6)}}{8}$$

$$y = \frac{-4x \pm \sqrt{16x^2 - 16x - 96}}{8}$$

$\therefore$ It is given that, y is real

$$\therefore 16x^2 - 16x - 96 \geq 0 \Rightarrow x^2 - x - 6 \geq 0$$

$$\Rightarrow x^2 - 3x + 2x - 6 \geq 0 \Rightarrow x(x-3) + 2(x-3) \geq 0 \Rightarrow (x-3)(x+2) \geq 0$$

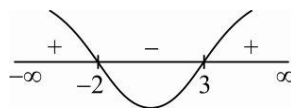
By wavy curve method, we get :

$$\Rightarrow x \in (-\infty, -2] \cup [3, \infty)$$

But it is given that $x \in (-\infty, a] \cup [b, \infty)$

$$\Rightarrow a = -2, b = 3$$

$$\therefore a + b = -2 + 3 = 1$$



152.(1) Given equation is well defined only for $-(x+3) \geq 0$

Let $k = -(x+3)$

$$\Rightarrow 3\sqrt{2k} - 1 = k + a \Rightarrow 3\sqrt{2k} = k + a + 1$$

$$\Rightarrow 18k = k^2 + (a+1)^2 + 2(a+1)k \Rightarrow k^2 + 2k(a-8) + (a+1)^2 = 0$$

For 2 distinct roots, the above quadratic equation must have 2 distinct roots ≥ 0

$$\Rightarrow D > 0, \text{ sum of roots } > 0, \text{ Product } \geq 0$$

$$D > 0 \Rightarrow (a-8)^2 - (a+1)^2 > 0 \Rightarrow -9(2a-7) > 0 \Rightarrow a < \frac{7}{2}$$

$$\text{Sum of roots} = -2(a-8) > 0 \Rightarrow a < 8$$

$$\text{Product} = (a+1)^2 \geq 0 \quad \forall a \in R$$

Hence, $a < \frac{7}{2}$ is the required condition

Maximum value of $[a] = 3$

153.(1) $b - c = \frac{1}{\log_2 2002} + \frac{1}{\log_3 2002} + \frac{1}{\log_4 2002} + \frac{1}{\log_5 2002}$

$$- \left(\frac{1}{\log_{10} 2002} + \frac{1}{\log_{11} 2002} + \frac{1}{\log_{12} 2002} + \frac{1}{\log_{13} 2002} + \frac{1}{\log_{14} 2002} \right)$$

$$= \log_{2002} 2 + \log_{2002} 3 + \log_{2002} 4 + \log_{2002} 5$$

$$- (\log_{2002} 10 + \log_{2002} 11 + \log_{2002} 12 + \log_{2002} 13 + \log_{2002} 14)$$

$$= \log_{2002} (2 \times 3 \times 4 \times 5) - \log_{2002} 11 \times 12 \times 13 \times 14$$

$$= \log_{2002} \left(\frac{2 \times 3 \times 4 \times 5}{10 \times 11 \times 12 \times 13 \times 14} \right) = \log_{2002} \frac{1}{2002} = -1 = |-1| = 1$$

154.(2) Given, $|x - k| < 2$ (i)

Solution set of equation (i) is subset of solution set of inequation,

$$\frac{2x-1}{x+1} < 1 \quad \text{.....(ii)}$$

From equation (i), we have

$$|x - k| < 2$$

$$\Rightarrow (x - k) > -2 \text{ and } (x - k) < 2 \quad [\because |x| < a \Rightarrow -a < x < a]$$

$$\Rightarrow x > k - 2 \text{ and } x < (k + 2) \Rightarrow k - 2 < x < k + 2$$

$$\Rightarrow x \in (k - 2, k + 2) \quad \text{.....(iii)}$$

From Equation (ii), we have

$$\frac{2x-1}{x+1} < 1$$

$$\Rightarrow \frac{2x-1}{x+2} - 1 < 0 \Rightarrow \frac{2x-1-(x+2)}{x+2} < 0 \Rightarrow \frac{2x-1-x-2}{x+2} < 0 \Rightarrow \frac{x-3}{x+2} < 0$$

Equating all the factors to zero, we get $x = 3, -2$

By wavy curve method, we have

$$x \in (-2, 3) \quad \text{.....(iv)}$$

Since, equation (iii) is subset of interval equation (iv)

$$\therefore k - 2 \geq -2 \text{ and } k + 2 \leq 3 \Rightarrow k \geq 0 \text{ and } k \leq 1 \Rightarrow 0 \leq k \leq 1$$

Hence, the integral values of k are 0 and 1 i.e. 2 integral values of k are possible.